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Measurement error models with uncertainty about the error variance

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Abstract

It is well known that measurement error in observable variables induces bias in estimates in standard regression analysis and that structural equation models (SEM) are a typical solution to this problem. Often, multiple indicator equations are subsumed as part of the SEM model – allowing for consistent estimation of the relevant regression parameters.

In many instances, however, embedding the measurement model into SEM is not possible because the model would not be identified. To correct for measurement error one has no other recourse than to provide the exact values of the variances of the measurement error terms of the model, although in practice such variances cannot be ascertained exactly but only estimated from an independent study. The usual approach so far has been to treat the estimated values of error variances as if they were known exact population values in the subsequent SEM analysis.

In this paper we show that fixing measurement error variance estimates as if they were true values can make the reported standard errors of the structural parameters of the model smaller than they should be. Inferences about the parameters of interest will be incorrect if the estimated nature of the variances is not taken into account.

For general SEM, we derive an explicit expression that provides the terms to be added to the standard errors provided by the standard SEM software that treats the estimated variances as exact population values. Interestingly, we find there is a differential impact of the corrections to be added to the standard errors depending on which parameter of the model is estimated. The theoretical results are illustrated with simulations and also with empirical data on a typical SEM model.

Key words: *Structural equation modeling, measurement error, reliability, correction for measurement error, fixed parameters, error variance, standard errors, standard error of reliability*

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Introduction

Measurement error in variables is a serious problem for the estimation of regression models (e.g. Fuller, 1987; Bollen, 1989). Error in dependent variables will bias R^2 and standardized regression coefficients, while error in independent variables will, in addition, bias unstandardized regression estimates. Ignoring such errors can severely affect estimates and conclusions. An important issue in the analysis of (systems of) regression equations is therefore how to correct for measurement error.

It is widely known that structural equation modeling (SEM) can deal with this issue. SEM allows the researcher to estimate simultaneous regression equations while also correcting for measurement error. Nowadays, the most common way to deal with measurement error is to consider multiple-indicator models in addition to the regression ("structural") part of the model (Bollen, 1989, chapters 5–6).

Figure 1(a) shows the simplest possible example of such a model: the latent independent and dependent variables of interest are respectively denoted ξ and η and their indicators x_1, x_2, y_1, and y_2. The arrows at the bottom signify the influence of measurement error. Estimating this model, the researcher obtains estimates of the variance of the measurement error in the indicators, as well as the "structural" relationship between ξ and η corrected for measurement error. Thus, the correction for measurement error is subsumed in a SEM model.

A second possibility, recommended in various textbooks on SEM (Bollen, 1989; Hayduk, 1987; Schumacker & Lomax, 2004), can be characterized as the "two-step" (TS) procedure. The latent variable ξ is defined as having the sum $x_1 + x_2$ as a single indicator, while η has the sum $y_1 + y_2$ as a single indicator. Means or weighted sums are also used as observed composite scores. The resulting model, shown in figure 1(b), is not identified. The variance of the measurement error in the composite scores must be calculated separately based on the error variances of the indicators

(Saris & Gallhofer, 2007), or obtained from other sources such as published reliability studies. Step one is then to fix the error variance parameters for the two composite scores to these estimates. In the second step the structural model is estimated while correcting for these fixed measurement error variances.

The TS solution to correction for measurement error has at least two advantages. First, it can dramatically reduce the size of the model and the number of possible parameters the researcher has to deal with. Since at least two indicators are usually needed for each latent variable to subsume the measurement error in the model, the number of variables is reduced by at least one half.

A second advantage of the TS method is a separation of labor between reliability studies and more substantive research. To estimate measurement error variances, one requires specific research designs that are adequate for this purpose. Often the models that must be applied to these designs are rather complex (Alwin, 2007; Saris & Gallhofer, 2007). To subsume the measurement error into the analysis, it is therefore necessary that the research design both address the substantive question and allow for the estimation of measurement error. It is also required that the researcher has both the substantive knowledge to formulate the correct structural model, and, in addition, is well-versed in the analysis of measurement error models. The TS solution allows substantive researchers to concentrate on the substantive model while still correcting for measurement error.

Examples of studies employing the TS approach abound in the literature. Some examples from different fields of application are Beckie and Hayduk (1997, 30), Varki and Colgate (2001, 236), Small et al. (2003, 170), and Rhodes et al. (2006, 3150). Each of these studies employs an estimate of the error variance from a previous study or separate analysis, and then sets the corresponding parameters fixed to these estimates.

Despite the above advantages of the TS method, it has a problem: the error variances are fixed to values which are assumed to be true when in fact they are only estimates. The variability in these estimates is not taken into account in the second step, where they are held fixed. Therefore, for all parameters of the model, the standard errors reported by typical SEM software may need correction; in a stricter sense, the reported standard errors are conditional on the variance of the estimated error variance being zero. To our knowledge, there is no report of the need for such corrections in published work using the TS approach.

A method to correct for the estimated nature of the variances of error terms in multivariate regression models is developed in Amemiya and Fuller (1984), and further discussed by Fuller (1987). To our knowledge, this solution has only been implemented in the software EV CARP (Schnell et al., 1987). The solution of Amemiya and Fuller (1984) is confined, however, to multivariate regression models and further assumes that the error variance estimates follow a chi square distribution with given degrees of freedom. In our approach, we address the problem for general SEM (including multivariate regression) without imposing a particular distribution on the estimate of error variance. Extensions for SEM in the case of categorical, count, or censored dependent variables and complex sampling designs will also be encompassed in our approach.

The remainder of the paper is structured as follows.

The next section presents a small simulation study in a very simple set-up that demonstrates why the estimated nature of the error variances should be taken into account. Section 3 presents an empirical example illustrating the TS analysis and the potential impact on inference of ignoring the estimated nature of fixed parameters. Section 4 derives an analytic expression for an additive correction term to the classical asymptotic standard errors. Section 5 presents a simple model set-up where the differential impact of the correction of different parameters of the model is illustrated. Section 6 is a Monte Carlo study that assesses the performance of the correction proposed. Finally, section 7 concludes.

The problem of uncertainty about the reliability estimates

Uncertainty about the reliability is a potentially important source of variability in estimates corrected for measurement error using the TS method. A short simulation shows why this should be the case, and why the uncertainty about measurement error should be taken into account.

One of the simplest and most well-known procedures of correction for measurement error is the classical correction for attenuation of the correlation coefficient (e.g. Fuller, 1987):

$$\tilde{\rho}(\eta, \xi) = \frac{\hat{\rho}(y, x)}{\kappa_1 \kappa_2}, \tag{1}$$

where y and x are observed indicators of respectively η and ξ; $\tilde{\rho}(\eta, \xi)$ and $\hat{\rho}(y, x)$ are respectively the estimates of the true correlations of η and ξ and y and x; and κ_1 and κ_2 are the corresponding reliability "coefficients" (Saris & Gallhofer, 2007) or "ratios" (Fuller, 1987) of y and x. We take $\hat{\rho}(y, x)$ to be the usual Pearson correlation. In practice, however, since κ_1 and κ_2 are not known, but only the estimated values $\hat{\kappa_1}$ and $\hat{\kappa_2}$ are available, the estimate of the correlation among η and ξ is simply

$$\hat{\rho}(\eta, \xi) = \frac{\hat{\rho}(y, x)}{\hat{\kappa}_1 \hat{\kappa}_2}. \tag{2}$$

We will now illustrate using Monte Carlo how the distribution of $\hat{\rho}(\eta, \xi)$ is affected by the variability of the estimates $\hat{\kappa_1}$ and $\hat{\kappa_2}$. In each simulation the correlation between the observed variables $\rho(y, x)$ is known and held constant at the true population value; but a random draw is taken from the sampling distribution of the reliability estimates $\hat{\kappa}_1$ and $\hat{\kappa}_2$. The resulting variation in the corrected correlation is therefore exclusively due to variation in the reliability estimates.

The correlation between the observed variables $\rho(y, x)$ was chosen to equal 0.40. The reliabilities κ_1^2 and κ_2^2 in the population were chosen as $\kappa_1^2 = 0.75$ and $\kappa_2^2 = 0.65$. These choices imply that in the population the true correlation between the two constructs of interest $\rho(\eta, \xi) \approx 0.573$. We assume the sampling distributions of the reliability estimates are normal with a certain standard error, and drew 500 samples from this distribution of $\hat{\kappa_1}^2$ and $\hat{\kappa_2}^2$. On average the estimates of $\hat{\kappa}_1^2$ and $\hat{\kappa}_2^2$ equaled their population values 0.75 and 0.65, but in any given sample the estimate differed from the population value due to variation in the reliabilities. For each sample drawn, we then compute the corrected correlation coefficient $\hat{\rho}(\eta, \xi) = 0.40/\left(\hat{\kappa}_1 \hat{\kappa}_2\right)$, obtaining 500 estimates of the correlation between the latent variables.

This process was repeated using different standard errors for the reliabilities. Increasing amounts of uncertainty about the reliability were used by setting the standard errors of the reliabilities to 0.001, 0.01, 0.05, and 0.1. Figure 2 shows the box-plots of 500 draws from four distributions of the corrected correlation for these different standard errors of the estimates of the reliability.

The dotted line in figure 2 shows that the mean of all of these distributions equals the true corrected correlation 0.57. It can be seen that all of the estimates are far away from the population correlation between the observed variables of 0.4; if the correction had not been made then a very precise but biased estimate would have been obtained.

When the standard errors of the estimated reliability are very small (.001), almost all of the corrected correlations are very close to the true correlation $.57$. Small standard errors of .01 already cause a greater variability in these corrected values. For medium and larger standard errors – of .05 and .10 – the corrected correlations vary considerably. For each Monte Carlo condition, the box-plot of the statistic $\hat{\rho}(\eta, \xi)$ shows that corrected correlations varied between .42 and .87 when the standard error was .10. Of the corrected correlations, 95% lay between .49 and .70.

This simulation shows that treating the reliability as a known constant is only warranted when it has been precisely estimated with a very small standard error. In general, the estimate of the true correlation can be affected quite a bit by uncertainty in the measurement error variance or reliability. Therefore the assumption of zero uncertainty about measurement error variances gives an over-optimistic assessment of the variability of the estimate of the true correlation.

Measurement error in structural equation models: an example

The previous section showed that uncertainty about the reliabilities can be a problem in the TS method of correction for attenuation of correlations and simple regression models.

The same problem of under-estimation of the variability of estimates of interest arises in more complex settings. This section illustrates, with empirical data and a simple model context, the issue of this bias when assessing regression effects in general SEM. The same model example will be used later for implementing the corrections developed in the paper.

This section uses empirical data to provide an example of a structural equation model with correction for measurement error. The model chosen contains reciprocal effects and correlated errors, making it a more general model than a multiple regression. After formulating the structural part of the model, the TS procedure of correction for measurement error is applied using previously obtained estimates of the variances of the error variables.

The relationship between political and social trust is a central point of interest in the study of social capital. While Putnam (2001) has argued that social trust engenders good government and subsequently trust in politics, others have argued that political trust may in turn affect social trust: the correlation between social and political trust may be due to effects in both directions. Saris and Gallhofer (2007) discuss an analysis of the reciprocal effect between social and political trust with correction for measurement error. See Saris and Gallhofer (2007) for a more substantive discussion of this model. A simplified version of their model is shown in figure 3.

The model shown in figure 3 contains a reciprocal effect between the variables “social trust” and “political trust”, as well as a covariance between the disturbance terms of these two variables. The exogenous variable “fear of crime” affects only “social trust” and not “political trust”, while “political efficacy” affects only “political trust” and not “social trust”. These restrictions are enough to identify the reciprocal effect between “social trust” and “political trust”, as well as the disturbance term covariance.

Whether this model is correct must be studied by careful examination of the underlying theory and the model fit to actual data, which is not the topic of the present example (see Saris & Gallhofer, 2007). Here we will discuss only how measurement error can affect the analysis of such a complex model.

All four variables in the model are complex concepts, measured as simple sum scores of several survey questions[1]. Data from the European Social Survey round 4 (2008) in Denmark are used. The sample size was 1610 Danish residents, surveyed by computer-assisted personal interviewing (CAPI) in their home. Table 1 shows the summary statistics for the resulting sum scores. Also shown is the estimated reliability for each sum score, and the corresponding estimate of the error variance, with standard errors[2]. This table also shows the variances of the estimates of reliabilities.

The model shown in figure 3 can easily be specified in standard structural equation modeling software[3]. The latent variables “fear of crime”, “political efficacy”, “social trust”, and “political trust” are each specified to have a composite (sum) score as a single indicator. The error variance ψ_{ii} of each single indicator is fixed to the corresponding value found in the last column of Table 1. The relationships between the latent variables are then estimated, correcting for measurement error in the composite scores.

Using this procedure we can obtain estimates of the “structural” parameters in the model corrected for measurement error. If the model is specified without latent variables or by fixing the error variances to zero, the “naive” parameter estimates without correction for measurement error are obtained. These estimates were obtained by using the `OpenMx` package in `R` (Boker et al., 2010; R Development Core Team, 2010). Table 2 shows both the naive and measurement error-corrected unstandardized estimates with standard errors and z-values.

Table 2 shows that in complex models such as the one analyzed, measurement error biases the estimates. This bias is not necessarily downwards. The corrected effects of the instruments efficacy and fear of crime are indeed increased after correction for measurement error, but the reciprocal effects between social and political trust are lower than without correction for measurement error. The standard errors for the corrected coefficients are larger, but those of the effects of efficacy and fear of crime are much more increased than the standard errors of the reciprocal effects of social and political trust.

The parameters of most interest to substantive researchers are the direct effects of social trust on political trust and vice versa, as well as the so-called ’total effects’. The direct effect of social trust on political trust is stronger than the converse effect. The total effects of social trust and political trust can be calculated as 1.0 and 0.40, respectively. This implies that for a given amount of change in social trust, political trust, which was measured on the same scale, is expected to increase by the same amount. The effect is also statistically significant at the 0.05 level ($z = 2.7$). The reverse is not the case, as social trust can be expected to increase by only 40% of the change in political trust, but this effect is not statistically significant at the 0.05 level ($z = 1.2$). This is largely in correspondence with suggestions made in the literature on social capital (Newton, 2007).

If the same model were analyzed *without* correction for measurement error, the conclusions would be rather different. Without correction, both direct effects are significant at the .05 level. The direct effect of social trust on political trust and vice versa are 11% and 50% too large respectively, and the total effects are respectively 40% and 50% overestimated.

It is clear that in the example, correction for measurement error has effects that do not just bias the results in a "conservative" direction, but that affect the conclusions in an way that is unpredictable without knowledge of the variable's reliabilities. For this reason, the estimation of reliability and correction for the errors is essential.

This example assumed that the error variances were known precisely, rather than estimated as was the reality. We will now show that inferences on parameters of interest may be affected.

Correction of the standard errors for uncertainty about fixed error variances

Given a vector of observed variables z with covariance matrix Σ, a structural equation model can be viewed as a covariance structure model $\Sigma = \Sigma(\theta)$, where $\Sigma(\theta)$ is a continuously differentiable matrix-valued (symmetric and positive definite) function of the vector of parameters θ of the model (see, e.g. Bollen, 1989, for more details).

Given an observed covariance matrix S, based on a sample of size n of z, the vector of parameters θ is estimated by minimizing with respect to θ a discrepancy function $F(\theta) = F(S, \Sigma(\theta))$ of S and $\Sigma(\theta)$. This approach encompasses maximum likelihood as well as weighted least squares estimators. Key matrices in deriving standard errors are the hessian matrix $V = \frac{1}{2}\frac{\partial^2 F(S,\Sigma)}{\partial\sigma\partial\sigma'}$, where σ is the half vectorization of Σ. In weighted least squares estimators, V is a weight matrix determined by the choice of estimator (Satorra, 1989).

In the specification of the model, we assume a matrix of variances and covariances of measurement errors to take the value Ψ. If there is no uncertainty about Ψ, a standard formula for the variance of the parameter estimates applies (Satorra & Bentler, 1990, 239). We will denote this variance as $\mathrm{var}_{\mathrm{standard}}(\hat{\theta})$. This is the standard error output given by SEM software. The precise form of this equation will depend on distributional assumptions as well as possible complex sampling and other issues, which are not the topic of this discussion. All forms of $\mathrm{var}_{\mathrm{standard}}(\hat{\theta})$ assume that the fixed measurement error variances in the matrix Ψ are exactly known with no uncertainty.

When the matrix Ψ of measurement error variances and covariances is not known exactly, but only fixed to a consistent estimate, the estimation procedure still provides consistent estimates corrected for measurement error. Standard errors obtained from $\mathrm{var}_{\mathrm{standard}}(\hat{\theta})$ must be adjusted according to the "uncertainty" in the Ψ matrix.

Let $\psi := \mathrm{vech}\,\Psi$, and denote as Σ_ψ the variance matrix of the estimation of ψ. We will now see how this term, Σ_ψ, when it is not equal to zero, affects the standard errors of estimates of all parameters of the model.

Consider the vector of parameter estimates $\hat{\theta}$, which is a function of the data, the observed covariance matrix S, and of the now random vector ψ.

$$\hat{\theta} = \hat{\theta}(S, \psi). \tag{3}$$

Conditioning on ψ the variance of the estimate $\hat{\theta}$ equals:

$$\mathrm{var}(\hat{\theta}) = E_\psi[\mathrm{var}(\hat{\theta}(S|\psi))] + \mathrm{var}_\psi[E(\hat{\theta}(S|\psi))], \tag{4}$$

where E_ψ and var_ψ denote conditional expectation and variance (conditional on given ψ). The first term will then be close to the "standard" variance formula,

$$E_\psi[\mathrm{var}(\hat{\theta}(S|\psi))] \approx \mathrm{var}_{\mathrm{standard}}(\hat{\theta}). \tag{5}$$

By a Taylor expansion, the second term of variance can be expressed as

$$\mathrm{var}_\psi[E(\hat{\theta}(S|\psi))] \approx \left(\frac{\partial\hat{\theta}}{\partial\psi'}\right)\Sigma_\psi\left(\frac{\partial\hat{\theta}}{\partial\psi'}\right)', \tag{6}$$

a term that involves the Jacobian matrix $\partial\hat{\theta}/\partial\psi'$ and the variance-covariance matrix Σ_ψ. This is clearly a non-negative definite matrix.

We see that the correct asymptotic variance of the estimated free parameter vector $\hat{\theta}$ under the null hypothesis is the sum of the right hand sides of equations 5 and 6. That is, the corrected variance is the sum of the 'standard' variance and the added variance due to a non-zero matrix Σ_ψ. This added variance, the correction term to be added, will be denoted C_ψ. This finding is important because it shows that the variance of the estimates of all parameters of the model is always increased by uncertainty about the fixed measurement error parameters ψ. Note that the term to be added, C_ψ, will be all-zero when the matrix Σ_ψ is all-zero.

We will now derive an explicit formula for the variance of parameters of structural equation models with uncertainty about ψ. Let $f(\hat{\theta},\psi) := \partial F/\partial\theta$. Then $\hat{\theta}$ and ψ are related by the fact that $\hat{\theta}$ equals the solution to the equation $f(\hat{\theta},\psi) = 0$. Invoking the implicit function theorem:

$$\frac{\partial\hat{\theta}}{\partial\psi'} = -\left(\frac{\partial f}{\partial\hat{\theta}}\right)^{-1}\frac{\partial f}{\partial\psi} = -\left(\frac{\partial^2 F}{\partial\theta\partial\theta'}\right)^{-1}\left(\frac{\partial^2 F}{\partial\theta\partial\psi'}\right). \tag{7}$$

From Satorra (1989), it can be seen that the asymptotic limits of $\frac{\partial^2 F}{\partial\theta\partial\theta'}$ and $\frac{\partial^2 F}{\partial\theta\partial\psi'}$ are respectively $J = \Delta_\theta' V \Delta_\theta$ and $H = \Delta_\theta' V \Delta_\psi$, letting $\Delta_\theta = \partial\sigma/\partial\theta'$ and $\Delta_\psi = \partial\sigma/\partial\psi'$, which expressions (for the LISREL model) can be obtained from Neudecker and Satorra (1991). So, using 6, the 'extra' term to be added to the variance of $\hat{\theta}$ due to the variance in estimating Ψ is

$$C_\psi = \mathrm{var}\left[E(\hat{\theta}(S \mid \psi))\right] \approx J^{-1}H\Sigma_\Psi H' J^{-1}. \tag{8}$$

With sample data, consistent estimates would replace the population matrices in the above expression. Note that the matrices Δ_θ and Δ_ψ depend only on model parameters and not on the estimation method. Thus, we have provided in equation 8 the expression that provides a one-step solution for the correction of the standard errors for the estimation of measurement error variances.

Features to note are that

1. The amount of variance to be added depends on the matrix Σ_ψ, the variance of the estimated measurement error variances;

2. The sample size is absent from the term to be added;

3. The term to be added will always be ≥ 0;

4. The correction is affected by the matrix H, which depends on the model structure. So it may highly differ for different model structures.

The next section will demonstrate the highly differential impact of the correction on different parameters in a simple model.

Illustration with a fictional regression model with uncorrelated regressors

In this section, an illustration using a fictional regression model will be presented. With this illustration we want to show how the corrections introduced should be used and to exemplify the differential impact on the corrections depending on which parameter of the model is considered.

Suppose the regression of a dependent latent variable η on two independent latent variables ξ_1 and ξ_2 is of interest. The dependent variable η is measured with an error-prone single indicator y, the independent variable ξ_1 with an error-prone single indicator x_1. The matrix Ψ corresponds to the variances of the measurement errors in these two indicators. The variance of the estimates of Ψ is the Σ_Ψ of the previous section. The independent variable ξ_2 is measured perfectly by the error-free observed variable x_2, so that we may just as well write x_2 instead of ξ_2. The two independent variables ξ_1 and x_2 are, moreover, uncorrelated with each other. This model can be formulated as a structural equation model, shown in figure 4.

The figure provides fictional population parameter values for the unstandardized coefficients. It can be seen that x_2 has been perfectly measured while x_1 and y both have a reliability of 0.75. The regression coefficients of interest β_1 and β_2 both equal 0.5, and the R^2 also equals 0.5. Importantly, the model has been formulated with uncorrelated independent variables.

Since x_2 is error-free, two estimated error variances have been fixed in the matrix Ψ. This measurement error variance matrix is not exactly known, but has been estimated with its own variance matrix Σ_ψ. The diagonal elements of this matrix, which denote the variance of the fixed error variance parameters, are named v_1 and v_2. The uncertainty about the error variance of the dependent variable y is called v_1 while v_2 is the uncertainty about the error variance of x_1.

Using equation 8 from the previous section, the correction term that should be added to the variance-covariance matrix of the parameter estimates when Ψ takes the values given in the figure is:

$$C_\psi = \begin{array}{c} \\ \sigma^2_\zeta \\ \sigma^2_\xi \\ \sigma^2_{x_2} \\ \beta_1 \\ \beta_2 \end{array} \begin{array}{c} \begin{array}{ccccc} \sigma^2_\zeta & \sigma^2_\xi & \sigma^2_{x_2} & \beta_1 & \beta_2 \end{array} \\ \left(\begin{array}{ccccc} v_1 + \frac{v_2}{16} & & & & \\ \frac{v_2}{4} & v_2 & & & \\ 0 & 0 & 0 & & \\ -\frac{v_2}{8} & -\frac{v_2}{2} & 0 & \frac{v_2}{4} & \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \end{array}, \tag{9}$$

where σ^2_ζ, σ^2_ξ, and $\sigma^2_{x_2}$ denote the variances of, respectively, the disturbance term in the dependent variable, the variance of the latent independent variable ξ, and the variance of the error-free independent variable x_2. The diagonal terms of the matrix C_ψ provide the corrections on the variance of $\hat{\theta}$ due to variance (uncertainty) in the given matrix Ψ. Each element in the matrix corresponds to a variance of a parameter or a covariance between two parameters. The parameters are shown in the row and column headers.

It can be seen that the standard error of the residual variance $\mathrm{var}(\zeta)$ is the most affected. The addition to the variance parameter of the independent variable ξ_1 is exactly equal to the variance of the error variance of its indicator x_1. The variance of the unstandardized regression coefficient β_1 is much (four times) less affected by uncertainty about the amount of measurement error than the other two parameters.

Remarkably, the variance of the regression coefficient β_2 and the variance parameter $\mathrm{var}(x_2)$, both parameters of the error-free variable x_2, are completely unaffected by the uncertainty about the error variances of y and x_1. Because the error-prone x_1 and error-free x_2 variables are uncorrelated, the derivative of the parameters β_2 and $\mathrm{var}(x_2)$ with respect to the two error variances equals zero. Since β_2 and $\mathrm{var}(x_2)$ are independent of the error variances of x_1 and y, their standard errors are immune to uncertainty about the error variances. While the situation of uncorrelated regressors is not often encountered in practice, it demonstrates the differential effect of uncertainty about the measurement error variances.

The diagonal of the matrix as a function of the square root of the non-zero elements of Σ_ψ is shown in figure 5. The increase in standard error of the parameter estimates will depend on the actual value of the "standard" variance of the parameter estimates.

Monte Carlo evaluation

This section uses Monte Carlo simulation to evaluate the performance of the proposed corrections. For the Monte Carlo we used the same model setting discussed in our earlier example analysis of political and social trust. The simulation employs a more realistic setting than our earlier simulation, as in that section the population parameter values were assumed known: here we will also simulate sampling variation in the parameter estimates.

We consider the model of figure 3, with the parameter values obtained from the model estimation on the Danish example presented earlier as true population values. The model has four error terms whose variances are provided by an external analysis. These are the variances for the errors in variables “socialTrust”, “systemTrust”, “fearcrime”, and “efficacy”. We recall that the estimates of variances provided by external analysis with the corresponding variances of these estimates are given in Table 1. From the parameter estimates and the error variances we computed a true covariance matrix Σ_0 to be used in the Monte Carlo study.

The Monte Carlo experiment considers variation in the uncertainty of the error variances. For each condition in the Monte Carlo design, we proceed as follows:

1. For each of the four error variances, 2000 replications of error variance values were obtained from a normal distribution with mean fixed to the error variances of Table 1 (last column of the table) and with a standard deviation equal to the standard deviations in brackets in the last column of Table 1 multiplied by a factor c that varies within the design of the Monte Carlo study. For each of the 2000 replications of error variances, we generated an independent sample of size n (to be varied in the study) from a multivariate normal distribution with covariance matrix Σ_0.

2. In each of the 2000 replications, we have an estimate of the error variance for each of the four error terms, and a sample of all the observable variables. We then used the TS approach to estimate the parameters of the model and the corresponding standard errors. We computed the uncorrected and corrected standard errors introduced in the section above. Here the uncorrected standard errors are the “naive” standard program output without taking the estimated nature of the error variances into account, while the corrected standard errors employ the correction proposed in the previous sections. In each replication, we also computed two types of 95% confidence intervals: one using the uncorrected standard errors, the other using the proposed correction.

3. For each condition of the Monte Carlo design, the percentage of samples out of 2000 for which the 95% confidence interval contained the true value was recorded.

The design is shown in Table 3, while the results for $c = 0$ are shown in Table 4. The main results of the simulation are presented in figures 6 and 7.

Note that in step 1, a scale value of $c = 0$ indicates no uncertainty, i.e. perfect estimates of the error variances, while a scale value of $c = 1$ indicates the same amount of uncertainty as was found in the example, and higher scale values indicate c times as much variation in the error variance estimate as found in our example. One consequence of this choice is that the different relative sizes of the standard errors will cause different effects on the standard errors of the model parameter estimates. This has been done in an attempt to introduce realistic differences in the relative amounts of uncertainty into the experiments. To provide a large range of different amounts of uncertainties, powers of two were taken as the scale values, yielding seven separate Monte Carlo conditions[4].

The scale values are essential in our simulations as they represent the actual amount of uncertainty present in the fixed parameters of the model for each experiment. The true population values of the error variances are shown in Table 3 along with their standard errors in the example (scale = 1). These values are difficult to interpret in absolute terms. Therefore table 3 also provides the corresponding true population reliability coefficients (i.e. the square roots of the reliabilities) and the standard errors of these reliability coefficients for different scale values[5]. Standard errors for $c = 0$ are not shown as they all equal zero. This provides an insight into the amount of uncertainty corresponding to each scale value. To get an impression of the relative size of the standard errors under different conditions, one can compare the sizes of the standard errors to those found on the horizontal axis of figure 2. It is clear from Table 3 and the preceding discussion that we can expect that the uncorrected standard errors will be more biased downwards as the scale factor increases, so that the coverage of 95% confidence intervals will worsen as the scale value becomes larger.

Without uncertainty in the measurement error estimates, the average of the simulations should equal the true parameter values and the coverage of 95% confidence intervals using both the uncorrected and the corrected standard errors should approach 95% as the number of simulations increases.

Table 4 shows the results of a Monte Carlo simulation without any uncertainty in the measurement error variances: in this experiment the assumption of perfect certainty about the error variances is correct. The table shows the average over samples of the parameter estimates $\bar{\theta}$, as well as the relative bias $(\bar{\theta} - \theta)/\theta$. The fourth column shows the standard deviation over samples $\mathrm{sd}(\hat{\theta})$

of the estimates. The subsequent two columns show the average of the uncorrected and corrected standard errors. As might be expected, these two are exactly equal when the assumption of perfect certainty is correct.

In each replication, we constructed two 95% confidence intervals for the parameter estimates: one using the uncorrected standard errors that one would obtain from standard software ($\text{C.I.}_{\text{uncor}}$), and one using the corrected standard errors proposed (C.I._{cor}). The last two columns show the percentage of replications for which the true parameter value is included in these two types of nominal 95% confidence intervals.

In general, the results shown in Table 4 suggest that the parameter estimates when there is no uncertainty about the fixed coefficients are approximately unbiased and that 95% confidence interval provide a coverage close to this nominal rate.

The real question, however, is what results are obtained when the assumption of perfect certainty about the error variances does not hold. We will now briefly discuss the simulation results. For the sake of brevity, here we give only the summary figures 6 and 7, which show the properties of the uncorrected and corrected standard errors at increasing levels of uncertainty[6].

Table 4 shows that uncertainty about the measurement error variance does not influence the unbiasedness of the estimates. In all simulations the estimates obtained are indistinguishable from those shown in the first column of Table 4, while also the relative bias is not affected. A slight difference starts to occur at very high levels of uncertainty due to an increased number of improper solutions. When these improper solutions are excluded from the analysis the effect disappears.

The standard deviation of the estimates across samples is, however, clearly affected by increasing uncertainty about the error variance parameters. Figure 6 shows, for each parameter of the model except ϕ_{21}, the standard deviation of that parameter's estimate across samples (the solid line marked "True"). It can clearly be seen that for all parameters the variability of the parameter estimate increases as the amount of uncertainty is increased.

The same figure also shows, in each graph, the average of the uncorrected standard errors and our corrected standard errors. The regular standard errors calculated under the assumption of perfect certainty about the error variances do not change appreciably as the uncertainty about the error variances increases. Since the “true” standard deviations across samples does increase for all parameters, this indicates an underestimation of the standard error. Our corrected standard errors do increase with the uncertainty, and follow the “true” standard deviation closely.

Figure 7 summarizes the main coverage results for all parameters. The graph on the left-hand side in figure 7 shows the proportion of 95% confidence intervals constructed using uncorrected standard errors (i.e. standard program output) that contain the true population parameter value. It can be seen that while some parameter estimates are not affected by the amount of uncertainty in this model, for other parameters the coverage properties worsen considerably. At the highest amount of uncertainty studied, the 95% confidence interval for the variance of “efficacy”, has deteriorated to a clearly unacceptable 40% coverage. Thus, the maximum deviation for all parameters from the target coverage of 95% without the correction proposed is 55%, while with the correction the largest deviation is only 3%. Other variance parameters are also affected, while unstandardized regression coefficients appear impervious in this model.

The right hand side of figure 7 shows the coverage of 95% confidence intervals constructed using our corrected standard errors. The graphs show that the correction term proposed here succeeds in correcting the standard errors of the estimates for uncertainty in the fixed error variance parameters.

Discussion and conclusion

Measurement error biases parameter estimates in structural equation models, necessitating corrections. SEM allows the researcher to subsume measurement error directly into the model, simultaneously estimating and correcting for measurement error.

However, this method requires that the study design is adequate both for the analysis of the substantive model and the evaluation of measurement error. It also requires that the researcher is knowledgeable about both measurement and substantive issues. A popular alternative that circumvents these disadvantages is to use estimates of measurement error variances from external studies to correct analyses. Error variance parameters in such analyses are fixed to the estimates that were obtained separately, followed by an analysis of the substantive model with correction for measurement error. We have called this method the “two-step” (TS) approach.

The TS approach does not take into account that the fixed error variances are, in reality, estimates with a certain amount of uncertainty. We have shown, first by a simple simulation and subsequently by analytics, that the uncertainty about fixed error variance parameters in the TS approach will bias standard errors downwards.

We presented an analytical solution to the problem of downwards biased standard errors in the TS method in the form of a term that should be added to the uncorrected covariance matrix of the parameter estimates. An estimate of this term can be readily calculated from the parameter estimates of the model[7].

The effect of uncertainty about the error variances, and therefore the downwards bias, depends on two factors: the *amount of uncertainty* and the *structure of the Jacobian matrix involving free parameters and fixed variances*. When there is no uncertainty (zero variance) about the fixed parameters, the corrected standard errors will equal the uncorrected standard errors.

However, even when there is uncertainty about the fixed parameters, there are models in which certain parameters’ variance will remain completely unaffected. This was demonstrated by the example of a multiple regression where the regressors are uncorrelated and one of the regressors is error-free. In such a case the parameters related to the error-free variable are completely independent of the measurement error variance of the error-prone variable, and uncertainty about the fixed error variance will not affect the unstandardized regression coefficient of the error-free variable. The independence in this model is created by the lack of correlation between the regressors.

The subsequent section evaluated the performance of the correction by Monte Carlo simulation. Simulations at different levels of uncertainty showed that the variability of the estimates can increase considerably due to uncertainty about the measurement error variance. It was shown that only our corrected standard errors provided the nominal confidence interval coverage, while uncorrected confidence intervals can deteriorate substantially in the presence of increasing uncertainty about the fixed parameters.

Generality of the solution The solution we have presented here applies to general structural equation models. As such, it encompasses a more general class of models than just multivariate regression. It also applies to different estimators in common use. Furthermore, multiple group SEM, growth curves, and categorical, count, or censored dependent variables can all be accommodated (Muthén, 2002), as well as complex sampling designs (Muthén & Satorra, 1995).

The solution has been motivated by, and formulated for, the case of fixed measurement error variance parameters. But it is not limited to only those types of parameters; variance in any vector of fixed parameters can be taken into account using this method.

One may wonder what happens when there is also uncertainty about the matrix Σ_{ψ}. In that case one could carry out a sensitivity analysis to examine the stability of inference on a certain parameter when the variance assigned to the estimation of ψ varies over a certain range.

Future studies The method presented was evaluated by Monte Carlo simulation using a specific model. However, a key aspect of the effect of uncertainty on a parameter as shown in equation 6 is that the parameter must be influenced by changes in the error variance. This was demonstrated explicitly by the numerical example of a multiple regression with uncorrelated regressors. This example could be constructed in such a way that some parameters are affected by the uncertainty about the error variances, while others are immune to this effect.

The amount of uncertainty was taken from a particular example and varied from that point. It was not shown that such degrees of uncertainty occur in practical research. However, there are indications that they do.

For example, an often-cited reliability study by Ware et al. (1978, pp. 40–42) on quality of life and subjective health indicators reports test-retest reliability coefficients between 0.4 and 0.7 based on a sample size of 138 persons. This suggests the standard errors for the reliabilities in this study are between approximately 0.04 and 0.06. The simulation shows that such standard errors are quite high and can have serious implications for inference. Other studies may employ a larger sample, but report separate coefficients for men and women of different age groups (e.g. in Lundberg & Manderbacka, 1996, n=204 and n=409). This will also cause the uncertainty in the reliability estimates to increase substantially.

These examples do not give a systematic study of the literature on reliability and as such cannot be generalized. However, it can not be ruled out that there are many cases in which the amount of uncertainty in reliability coefficients or error variance parameters, combined with the form of the model and the parameters of interest (e.g. standardized or unstandardized regression coefficients), cause the correction we propose to be necessary for correct inference.

In short, the strength or existence of an effect depends on the model used as well as the amount of uncertainty. The evaluation provided in this paper is therefore limited in that we do not know whether the characteristics of the model chosen for the evaluation are typical of applied work. This is clearly an important topic for further research, as it determines under which conditions the effect of uncertainty about error variance is a problem for practitioners.

Without more detailed knowledge about these conditions one should not discount the possibility that inference is affected. Therefore whenever measurement error variances have been fixed to an estimate about which uncertainty exists, the correction to standard errors presented here should be applied.

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Covariance matrix used in the example and Monte Carlo simulation

The example analysis was based on the following observed covariance matrix, which was also used as a population covariance matrix to generate samples for the Monte Carlo simulation.

	socialTrust	efficacy	systemTrust	fearcrime
socialTrust	22.38			
efficacy	0.86	2.81		
systemTrust	9.32	1.50	23.5	
fearcrime	-1.63	-0.60	-1.6	3.0

Footnotes

[1]For the full questionnaire we refer to http://ess.nsd.uib.no/ess/round4/fieldwork.html

[2]The error variances and reliabilities were obtained by first fitting a confirmatory factor model to the indicators of these constructs. The error variance and reliability of the simple sum score was then obtained by adding “ghost variables” to the model. The standard errors of these quantities were obtained by bootstrapping (Raykov, 2009).

[3]We have used the `OpenMx` package in `R` (Boker et al., 2010; R Development Core Team, 2010), and LISREL (Jöreskog & Sörbom, 1996) to double-check the results.

[4]All analyses were carried out using `R` versions 2.10 and 2.11 for linux and the `OpenMx` package (R Development Core Team, 2010; Boker et al., 2010). The `R` code is available from the first author upon request.

[5]In calculating these standard errors the estimation of a covariance matrix from a sample of size 1500 was also taken into account.

[6]Full details of the Monte Carlo results are available upon request from the first author.

[7]An implementation for the SEM package `OpenMx` in `R` is available upon request from the first author.

Variable	Scale	Mean	Std dev	$\widehat{\mathrm{Rel.}}$ (s.e.)	$\hat{\psi}$ (s.e.)
fearCrime	3–12	5	1.7	0.57 (0.02)	1.3 (0.04)
efficacy	2–10	7	1.7	0.64 (0.03)	1.2 (0.07)
socialTrust	0–30	20	4.7	0.73 (0.01)	6.0 (0.22)
systemTrust	0–30	21	4.8	0.77 (0.01)	6.3 (0.24)

Table 1: Summary statistics for the Denmark dataset.

	Uncorrected			Corrected		
	Est.	s.e.	z	Est.	s.e.	z
β_{43}	0.86	(0.15)	5.9	0.77	(0.16)	4.9
β_{34}	0.45	(0.15)	3.0	0.30	(0.19)	1.6
β_{42}	0.27	(0.09)	3.1	0.51	(0.16)	3.2
β_{31}	-0.30	(0.10)	-3.1	-0.68	(0.22)	-3.0

Table 2: Unstandardized estimates without and with correction for measurement error. Only the regression coefficients in the model and not the variance parameters are shown for the sake of brevity.

				$\sigma(\widehat{\text{Rel}})$ for scaling factor c					
	$\hat{\psi}$	v_ψ	$\widehat{\text{Rel.}}$.	1	2	4	8	16	32
socialTrust	6.00	0.0484	0.85	0.01	0.01	0.02	0.05	0.09	0.19
systemTrust	6.29	0.0576	0.88	0.01	0.01	0.02	0.05	0.10	0.19
fearcrime	1.33	0.0016	0.75	0.01	0.02	0.04	0.07	0.15	0.25
efficacy	1.17	0.0049	0.80	0.02	0.03	0.07	0.14	0.24	0.33

Table 3: For the four variables of the model affected by measurement error, listed in the first column of the table, column two shows the estimated error variances, column three the variances of the estimated error variances, and column four the estimated reliability. To give an impression of the amount of uncertainty in the error variances under different conditions of the Monte Carlo design, columns 5 to 10 show for different scaling factors c the size of the uncertainty in terms of the standard error of the reliability.

	$\bar{\theta}$	Rel. bias	$\text{sd}(\hat{\theta})$	$\overline{\text{s.e.}}_{\text{uncor}}$	$\overline{\text{s.e.}}_{\text{cor}}$	$\text{C.I.}_{\text{uncor}}$	C.I._{cor}
β_{43}	0.77	-0.00	0.16	0.17	0.17	95.9%	95.9%
β_{42}	0.52	0.03	0.15	0.16	0.16	97.0%	97.0%
β_{34}	0.28	-0.08	0.20	0.21	0.21	98.5%	98.5%
β_{31}	-0.70	0.04	0.23	0.24	0.24	98.0%	98.0%
ϕ_{33}	12.19	0.06	3.87	1.96	1.96	97.0%	97.0%
ϕ_{22}	1.64	-0.00	0.10	0.10	0.10	94.3%	94.3%
ϕ_{34}	-6.22	-0.05	3.12	3.38	3.38	96.9%	96.9%
ϕ_{44}	12.42	0.03	1.60	1.68	1.68	94.4%	94.4%
ϕ_{12}	-0.60	0.00	0.07	0.08	0.08	95.4%	95.4%
ϕ_{11}	1.71	-0.00	0.11	0.11	0.11	95.0%	95.0%

Table 4: Summary of Monte Carlo condition with $c = 0$ (zero uncertainty in the estimates of measurement error variance). The first two columns show the average estimates $\hat{\theta}$ over 2000 replications, and the difference between this average and the true values. The third, fourth, and fifth columns show the standard deviation observed over replications of the estimates, the uncorrected standard errors, and the corrected standard errors. The last two columns show, for uncorrected and corrected standard errors, the percentage of replications in which 95% confidence intervals, constructed using the standard errors, contained the true value.

Figure Captions

Figure 1. Two different ways of correcting for measurement error in a simple regression using SEM

Figure (a). Simple regression with multiple indicators.

Figure (b). Regression with single indicators.

Figure 2. Estimates of the true correlation of interest can vary widely when variability in the reliability is medium or large. Each box represents 500 draws from the distributions of the two reliabilities for increasing standard errors. The sample size used to estimate the correlation between the observed variables is assumed to be large, so that the box-plots represent only variation due to the uncertainty about the reliabilities.

Figure 3. Structural equation model adapted from Saris & Gallhofer (2007a). Variance parameters are shown as circular arrows.

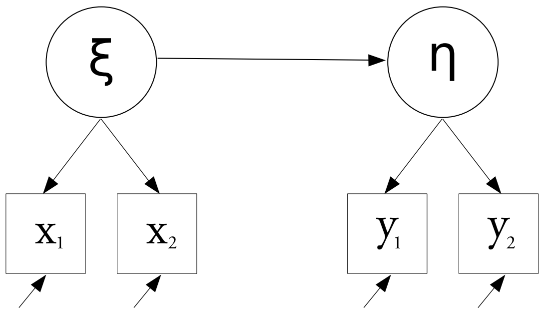
Figure 4. Path diagram for a multiple regression model. The independent variables ξ_1 and x_2 are uncorrelated. Circular arrows denote variance parameters. Unstandardized population parameter values are given. The reliabilities of x_1, x_2, and y equal 0.75, 1, and 0.75, respectively.

Figure 5. Increase in the variance of the three free parameter estimates as a function of the standard error of the measurement error variance (fixed) parameters.

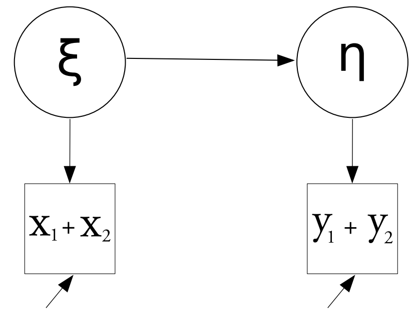
Figure 6. True standard deviations across the simulations (solid lines), corrected standard errors (striped lines), and uncorrected standard errors (dotted lines) for different uncertainty scale values. Each graph represents these relationships for one model parameter.

Figure 7. Coverage of nominal 95% confidence intervals for the different parameters of the model, as a function of the amount of uncertainty in the measurement error variance estimates. Left the coverage using the uncorrected (“naive”) standard errors is shown, while the right graph shows the

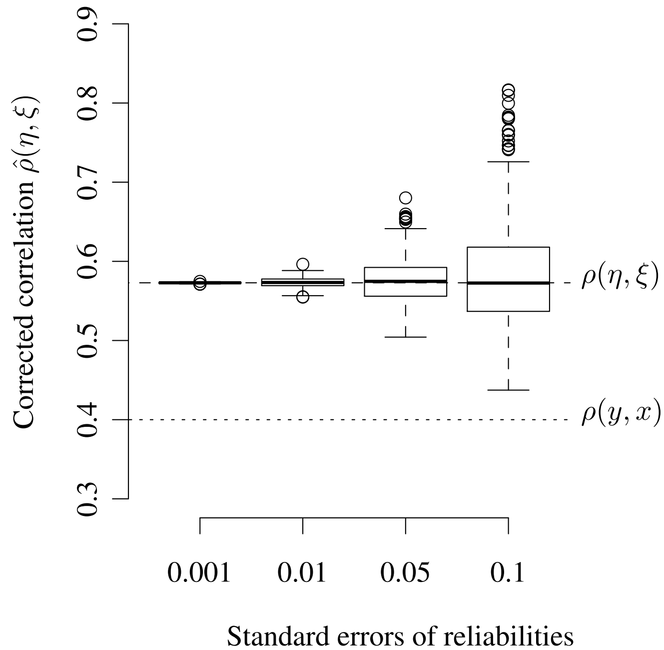
coverage when using the newly proposed corrected standard errors.



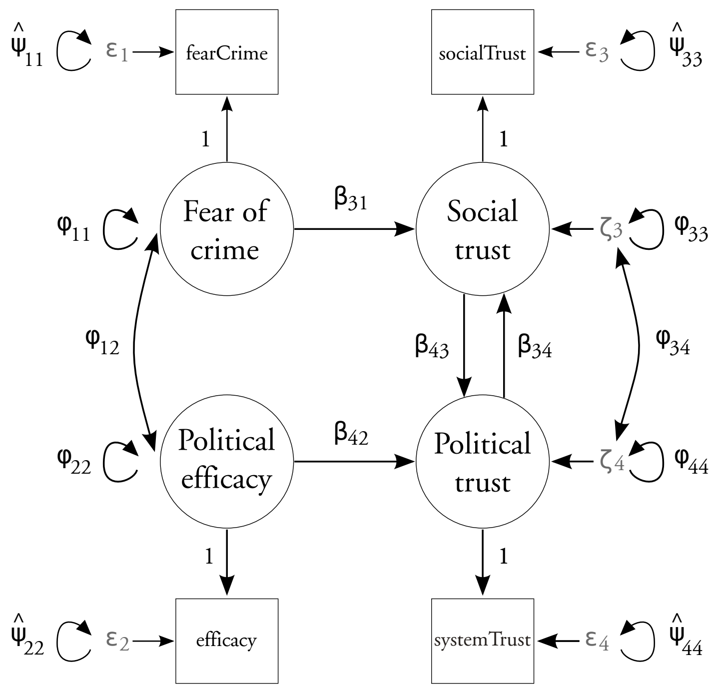
(a) Simple regression with multiple indicators.



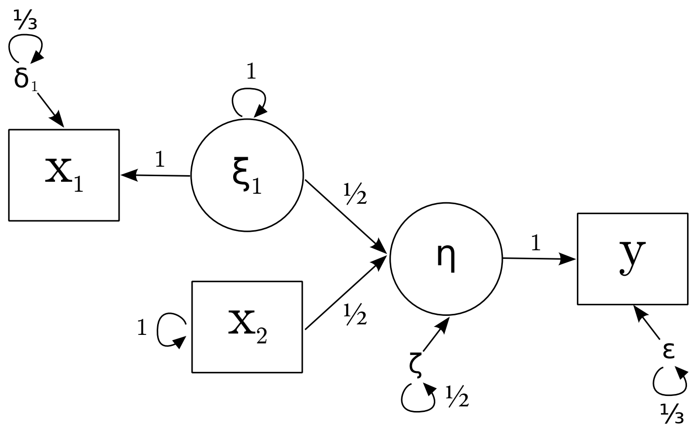
(b) Regression with single indicators.



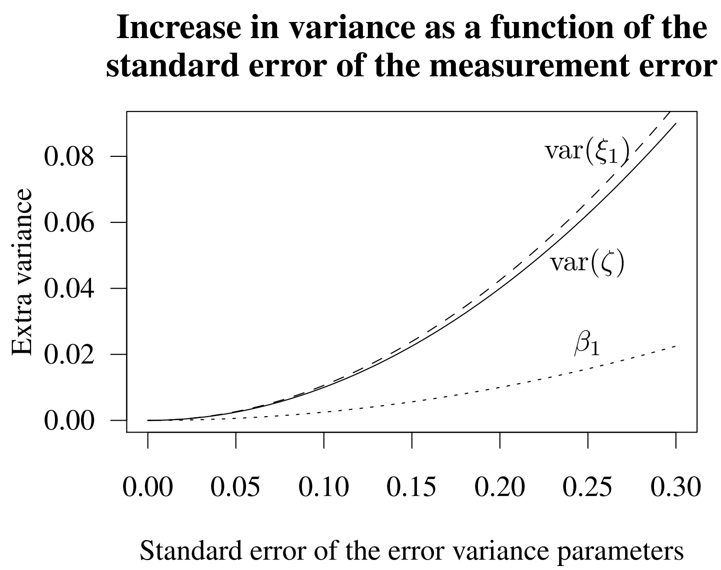
Corrected correlation $\hat{\rho}(\eta, \xi)$
0.3
0.4
0.5
0.6
0.7
0.8
0.9
$\rho(\eta, \xi)$
$\rho(y, x)$
0.001
0.01
0.05
0.1
Standard errors of reliabilities

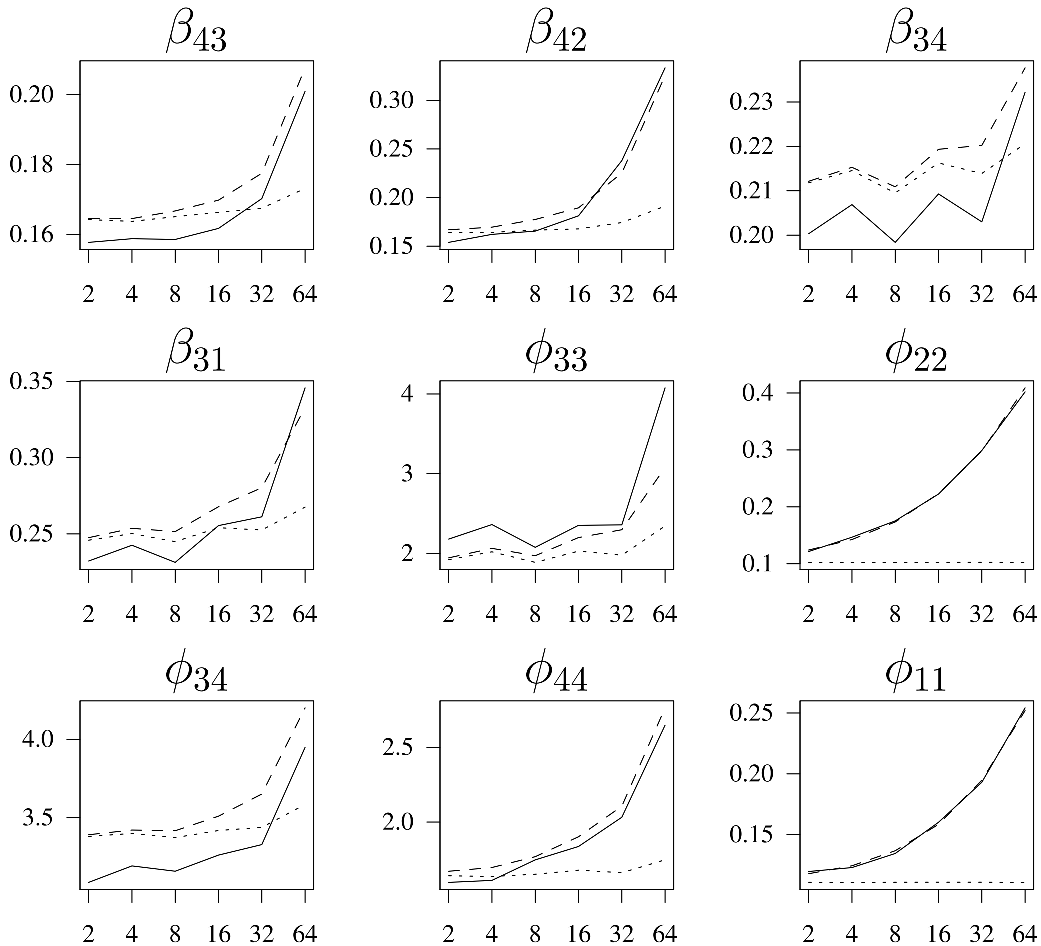


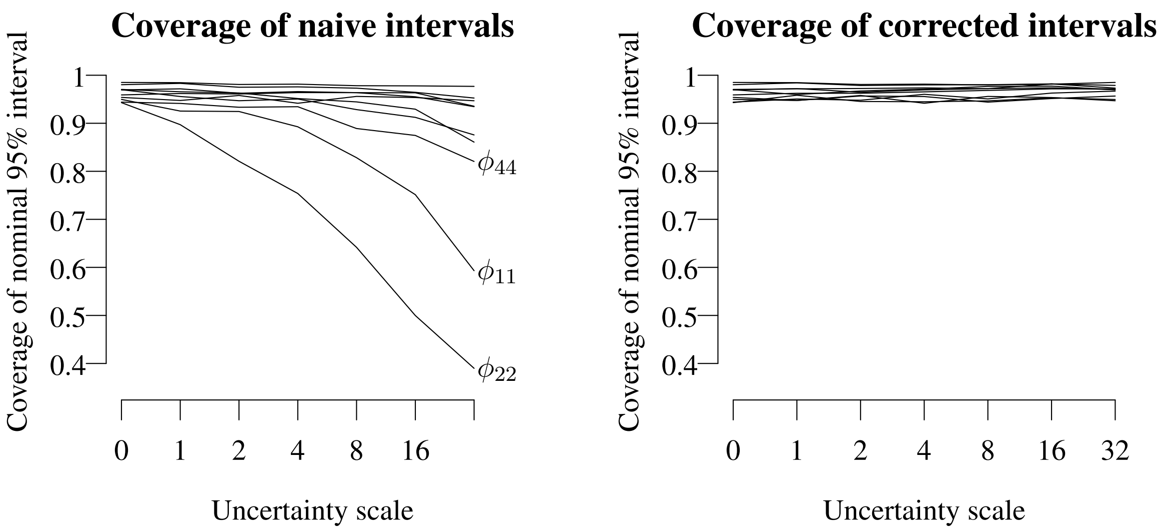
fearCrime
socialTrust
Fear of crime
Social trust
Political efficacy
Political trust
efficacy
systemTrust
1
1
1
1
β_{31}
β_{43}
β_{34}
β_{42}
φ_{11}
φ_{12}
φ_{22}
φ_{33}
φ_{34}
φ_{44}
ζ_3
ζ_4
ε_1
ε_2
ε_3
ε_4
$\hat{\psi}_{11}$
$\hat{\psi}_{22}$
$\hat{\psi}_{33}$
$\hat{\psi}_{44}$



⅓
δ1
1
x1
1
ξ1
½
η
1
y
1
x2
½
ζ
½
ε
⅓


Increase in variance as a function of the
standard error of the measurement error
Extra variance
0.00
0.02
0.04
0.06
0.08
0.00
0.05
0.10
0.15
0.20
0.25
0.30
Standard error of the error variance parameters
var(ξ_1)
var(ζ)
β_1


β_{43}
β_{42}
β_{34}
β_{31}
ϕ_{33}
ϕ_{22}
ϕ_{34}
ϕ_{44}
ϕ_{11}
0.16 0.18 0.20
0.15 0.20 0.25 0.30
0.20 0.21 0.22 0.23
0.25 0.30 0.35
2 3 4
0.1 0.2 0.3 0.4
3.5 4.0
2.0 2.5
0.15 0.20 0.25
2 4 8 16 32 64


Coverage of naive intervals
Coverage of nominal 95% interval
1
0.9
0.8
0.7
0.6
0.5
0.4
ϕ_{44}
ϕ_{11}
ϕ_{22}
0
1
2
4
8
16
Uncertainty scale
Coverage of corrected intervals
Coverage of nominal 95% interval
1
0.9
0.8
0.7
0.6
0.5
0.4
0
1
2
4
8
16
32
Uncertainty scale